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Original Article

Recording and Representing Student Mathematical Thinking: A Comparison of Preservice Teachers in the U.S. and Korea

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Abstract

The practice of recording and representing mathematical thinking of students is critical for enriching their understanding of mathematical concepts and procedures. However, teacher education often overlooks this competency, leaving it to be learned through experimentation in the field. This study aims to examine how mathematics preservice teachers (PSTs) perform in recording and representing mathematical ideas of students, offering insights to teacher educators to support PSTs' development of these skills. We compared PSTs in the U.S. and Korea in these teaching skills to reveal the intricate process involved. Ninety PSTs, 45 from each country, participated in representing and recording four different student mathematical strategies for two-digit addition problems. To examine PSTs' work in recording and representing student thinking, this study utilized an analytical framework consisting of five categories: authenticity, accuracy, organization, support for understanding, and specification. The findings indicate that both groups demonstrate higher performance in authentic translation of student work and accurate use of mathematical notation and values. However, both groups of PSTs show little evidence in their performance of representing mathematical thinking of students using a variety of visual models. We also found discrepancies between these two groups of PSTs. PSTs in Korea are more likely to record mathematical thinking as they are, while PSTs in the U.S. describe the student thinking pathways in more detail. This study has implications for mathematics teacher educators in both countries, highlighting specific elements they need to focus on to nurture PSTs' recording and representing of mathematical thinking of students.

Keywords: comparative study, core teaching practices, practice-based teacher education, preservice teacher education, recording and representing student thinking

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I. INTRODUCTION

The work of teaching, which refers to "the core tasks that teachers must execute to help pupils learn" (Ball & Forzani, 2009, p. 497), has gained recognition for its importance. Since teacher instructional practices are crucial to student mathematics learning, the quality of these practices influences students' mathematics achievement (National Council of Teachers of Mathematics [NCTM], 2014). Previous literature suggests that teacher preparation programs should offer preservice teachers (PSTs) more explicit opportunities to engage in crucial teaching practices (Ball & Forzani, 2009; Grossman et al., 2009; Lampert, 2010; Lampert & Graziani, 2009; NCTM, 2014). However, shifting "from a theory- and knowledge-based teacher education curriculum to one focused on practice is a complex undertaking" (Ball & Forzani, 2009, p. 506). To support this shift, educators have identified sets of teaching practices[1)] referred to as "core" (e.g., McDonald et al., 2013; Shaughnessy & Boerst, 2018) or "high-leverage" (e.g., Ball et al., 2009; TeachingWorks, 2024). While sets of core practices[2)] vary by proposing institutions and organizations (Grossman, 2018), core practices of teaching commonly represent the fundamental aspects of teaching aimed at fostering student learning. PSTs are expected to cultivate and apply these practices throughout their teacher training program and demonstrate a proficient level of mastery from the outset of their teaching careers (Association of Mathematics Teacher Educators, 2017; Ball et al., 2009; Davis & Boerst, 2014).

The set of core practices of teaching may sound so natural because most teachers implement some of these strategies in their classrooms (e.g., eliciting student thinking, leading a group discussion, and explaining and modeling content). However, each core practice consists of complex components that require breaking them into a set of parts (Grossman, 2011). For example, while leading a group discussion, the teacher needs to orchestrate multiple practices such as eliciting student thinking, orienting students to the ideas of their peers, maintaining the instructional focus, and recording and representing student thinking. These decomposed practices should be taught, observed, and practiced during teacher education (Grossman, 2011).

Among these various practices, this study focuses on recording and representing student thinking. Recording and representing student thinking on the board "can enable students to connect oral language, standard mathematical notation, and visual and physical representations in ways that can deepen conceptual understanding" (Garcia et al., 2021, p. 927). This practice is essential when teachers carry out core practices such as leading a discussion (e.g., Ball & Forzani, 2009; Forzani, 2014; Shaughnessy et al., 2021).

The practice of recording and representing student thinking is an intricate process. It involves eliciting

1) The list of high-leverage practices is one of the well-known sets of core practices of teaching proposed by TeachingWorks (2024). Although "core" typically denotes general teaching practices and "high-leverage" emphasizes practices deemed valuable for PSTs to acquire (Charalambous & Delaney, 2020), we consistently use core practices hereafter in this paper unless the term "high-leverage practices" is specifically cited from certain documents and studies.

2) A couple of examples include TeachingWorks' high-leverage practices and the core practices of the University of Washington's U-ACT program. Refer to the Appendix in Grossman (2018) to see various sets of core practices with different levels of specificity proposed by different institutions.

and interpreting student thinking, understanding mathematical representations, and deciding what to write or draw and how to establish connections (Grossman et al., 2009; NCTM, 2014; TeachingWorks, 2024). Despite the increased attention given to research on core practices, there has been limited research focused on recording and representing student thinking. Thus, this study draws on the written task, which invited PSTs to record and represent mathematical thinking of students regarding the addition of two-digit numbers (46 + 38), assuming that they are leading a classroom group discussion.

To identify any unique patterns of PSTs, we examined the work of PSTs in both the U.S. and Korea. As international comparative studies help researchers to understand the 'hidden national characteristics' (Blömeke & Paine, 2008) and the two countries have used each other as reference models (Flavin & Hwang, 2024; Kim et al., 2011), examining PSTs in both countries may help us understand the strengths and weaknesses of their knowledge and how to prepare them for teaching mathematics.

Our study aims to provide insights into how to support PSTs of each country to enhance their competency in recording and representing mathematical thinking of students. Specifically, the following research questions guided this study:

1. To what extent do PSTs record and represent mathematical thinking of students regarding the addition of two-digit numbers (46 + 38), and what attributes are identified in their work?
2. What are the commonalities and distinctions between PSTs in the U.S. and Korea in their work of recording and representing student mathematical ideas?

II. THEORETICAL FRAMEWORK

We frame our study with pedagogies of practice framework (Grossman et al., 2009) and professional noticing framework (Jacobs et al., 2010). Using these two frameworks, we situate the practice of recording and representing student thinking as a part of practice-based teacher education and a way for PSTs to make instructional decisions to support students' mathematical thinking.

1. Practice-based teacher education

Practice-based teacher education (PBTE) emphasizes the development of highly skilled teachers through core practices directly related to the work of teaching (Forzani, 2014). These core practices include instructional strategies and their subcomponents of routines and moves that are frequently used across various curricula and have shown potential to enhance student learning (Grossman, 2018; Grossman & McDonald, 2008). The primary objective of PBTE is to bridge the gap between theoretically driven teacher education programs and practical teaching work. It posits that core practices can be taught and that novice teachers can begin to master these practices while maintaining the integrity and complexity of teaching (Ball & Forzani, 2009;

Grossman et al., 2009).

Earlier attempts at PBTE often presented a limited perspective on teaching skills that lacked strong connections to research, leading to a simplistic understanding of what constitutes effective teaching (Zeichner, 2012). In contrast, contemporary PBTE models frequently adopt Grossman et al.'s (2009) pedagogies of practice framework, which delineates three essential concepts for professional education: representations of practice, decomposition of practice, and approximations of practice.

Representations of practice are forms used to depict aspects of practice for learners (i.e., PSTs in the context of teacher education) and involve tools that effectively portray the practice of professionals. These representations encompass a broad range of media, such as video recordings of teaching, written vignettes, and examples of student work, among other forms. These representations can be decomposed for the purpose of educating novice professionals. *Decomposition* of practice refers to "breaking down complex practice into its constituent parts for the purposes of teaching and learning" (Grossman et al., 2009, p.2069). This practice allows learners to focus on specific components or concepts. For instance, a teacher educator might highlight how a teacher records and represents student thinking during a group discussion. By doing this, PSTs can concentrate on the specific use of recordings and representations while temporarily setting aside other aspects of the teacher's practice.

Approximations of practices mean the "opportunities for novices to engage in practices that are more or less proximal to the practices of a profession" (Grossman et al., 2009, p.2058), varying on the continuum of less to more authentic opportunities. The level of authenticity in the approximation of practice depends on how closely it mirrors actual practice in various aspects, such as the demands on the teacher, the classroom environment, and the participants involved. For instance, a less authentic approximation of teaching practice could involve a novice teacher planning a lesson for a hypothetical group of students. In this case, the lesson development still includes some authentic practices, such as selecting or modifying instructional activities. However, this approximation involves carrying out lesson planning without knowledge of the actual students or classroom environment. Approximations of practices can enable novices to test the waters under simpler and less risky conditions (Grossman et al., 2009).

When designing a teacher education program based on the PBTE framework (Grossman et al., 2009), it is important to consider how to offer opportunities for PSTs to learn the work of teaching (i.e., representations of practices), to break down complex teaching into more manageable pieces (i.e., decomposition of practices), and to provide opportunities to enact core practices with authenticity (i.e., approximations of practice). While all three practices are important, our study focused on the decomposition of practices to examine how PSTs record and represent students' mathematical thinking when assumed to lead a classroom discussion.

2. Professional teacher noticing

To make appropriate instructional decisions that support students' learning, teachers must learn and practice intentional noticing with specific goals. Literature identifies professional teacher noticing as a crucial instructional practice that PSTs should develop in their education (Jacobs et al., 2010; Mason, 2002; Miller, 2011; Sherin & van Es, 2009). Mason (2002) introduced the concept of noticing and described it in two main categories: *accounts-of* and *accounts-for*. Accounts-of are purely descriptive observations of phenomena without interpretation or evaluation, representing the "what" dimension of noticing. Accounts-for involve exploratory and interpretive observations, representing the "how" dimension.

Similarly, Sherin and van Es (2009) conceptualized a two-part framework for teacher noticing, consisting of what teachers perceive and how they understand it. Jacobs et al. (2010) expanded the construct of noticing and developed a framework for a specialized form of teacher noticing, known as *professional noticing of children's mathematical thinking*. Jacobs et al. (2024) recapped the three interrelated component skills based on their earlier work as follows:

- *Attending to children's strategy details*: Teachers' recognition of mathematically noteworthy aspects of children's strategy while gaining a more nuanced view of children's thinking by going beyond the answer and focusing on multiple strategy details.
- *Interpreting children's understandings*: Teachers' reasoning about strategy details to discern children's mathematical understandings.
- *Deciding how to respond on the basis of children's understanding*: Teachers' determination of the following (intended) instructional steps describing how teachers use what they have learned from children's strategy details.

Jacobs et al. (2024) assert that these three skills are "conceptually and temporally linked, occurring almost simultaneously in the midst of instruction" (p. 297). However, as we focused on PSTs' work of using students' strategies presented in written texts, not actual instructional practices in authentic classrooms, this study highlights the interpreting component of the professional noticing framework.

III. SITUATING THE STUDY

1. Focusing on "Recording and representing" student mathematical ideas

Currently, no one set of core teaching practices exists. The sets of core practices vary by proposing institutions and specific disciplines (see the Appendix in Grossman (2018) for various sets of core practices with different levels of specificity proposed by different institutions and organizations). Also, maintaining a sense of grain-size among different core practices is challenging. For example, high-leverage practices, one of the well-known sets of core practices (TeachingWorks, 2024), include 19 core practices. Of the 19 core practices, "eliciting and interpreting individual student thinking" and "leading a discussion" are presented as two independent core practices. However, when examining the decomposition of leading a discussion as shown in Table 1, "eliciting" multiple ideas and "probing" students' thinking are embedded. This implies that the core practices have inherently overlapping characteristics that rely on one another to be successfully implemented (Nelson et al., 2022).

Table 1 *Decomposition of leading a discussion (adapted from TeachingWorks, 2024)*

Discussion Enabling	Discussion Leading		
• Selecting a task • Anticipating student thinking • Setting up the task • Monitoring student work	Framing – Launching	Orchestrating – Eliciting – Orienting – Probing – Making contributions	Framing – Concluding
	Recording and representing content		
	Maintaining a focus on the instructional point		
	Seeing and disrupting patterns that reproduce inequity		

Note. 'Recording and representing content' is highlighted in yellow to emphasize the focus of this study.

Similarly, our study recognizes that recording and representing student mathematical ideas is an essential component in the context of leading a group discussion (see Table 1). As a nested component of a bigger grain-sized core practice of leading a discussion, very few studies have been conducted focusing on PSTs' work of recording and representing student ideas. Two exceptions are Garcia et al. (2021) and Shaughnessy et al. (2021).

Garcia et al. (2021) mainly focused on the practice of recording and representing student thinking while facilitating a group discussion. They claimed that the way that teachers record student thinking positively impacts mathematical understanding of students because it can advance mathematical ideas, respect students as sense makers, and attend to the integrity of mathematics. When teachers record and represent student ideas, they may provide additional information or modify them to make the student's thinking more visible and more understandable to other students (Garcia et al., 2021). These include marking, annotating, and using visual cues and representations. Thus, teachers are expected to perform specific tasks when recording and representing student mathematical ideas. Those specific tasks include determining what content to record and in how much detail, as well as deciding whether to use additional language or other forms of representations showing students' ideas to support student understanding and classroom participation (TeachingWorks, 2024).

Similarly, Shaughnessy et al. (2021) emphasized four key elements of the execution of recording and representation practices during the classroom discussion, stating, "Recording student ideas in ways that are true to student contributions, attending to the accuracy of the records and representations, recording in ways that are clear, organized, and visible to the class, [and] using representations that support student understanding" (p. 460).

Table 2 briefly illustrates elements highlighted in the previous studies and how these key elements allow our study to examine PSTs' recording and representing mathematical thinking of hypothetical students.

Table 2 *Analytical framework for the current study and its relation to the prior studies*

Analytical framework from the prior studies			Analytical framework for the current study
Garcia et al. (2021): Principles for recording that support mathematical understanding	Shaughnessy et al. (2021): Four elements of skillful use of recording and representations	TeachingWorks (2024):	
1. Advance mathematical ideas (record core ideas, manage level of detail, separate key ideas and arrange them to support connections) 2. Respect students as sense makers (record what students say, add details to support sense making, label ideas to support discourse) 3. Attend to the integrity of mathematics (make thinking visible, add details to support students in remembering key decisions).	4. Recording student ideas in ways that are true to student contributions 5. Attending to the accuracy of the records and representations 6. Recording in ways that are organized and visible 7. Using representations that support student understanding	8. Determining what content to record and in how much detail 9. Deciding whether to attach student names to ideas 10. Attending to the accuracy of records and representations 11. Deciding when to use student informal language and when to support the use of conventional academic language 12. Recording in ways that are clear, organized, and visible to the class 13. Using language and/or representations that support student understanding and participation	A. Authenticity (related to 2, 4) A1. Authenticity in translation A2. Authenticity in student thinking paths B. Accuracy (related to 3, 5, 10) B1. Accuracy in notation (related B2. Accuracy in mathematical value C. Organization (related to 6, 8, 12) C1. Clarity and Flow C2. Consistency in the use of notation D. Support for understanding (related to 1, 7, 13) D1. Added details D2. Visual representations E. Specification E1. Explanations E2. Naming

Note. Category E was added to analyze parts of our study's data in which PSTs explained and named their recording decisions.

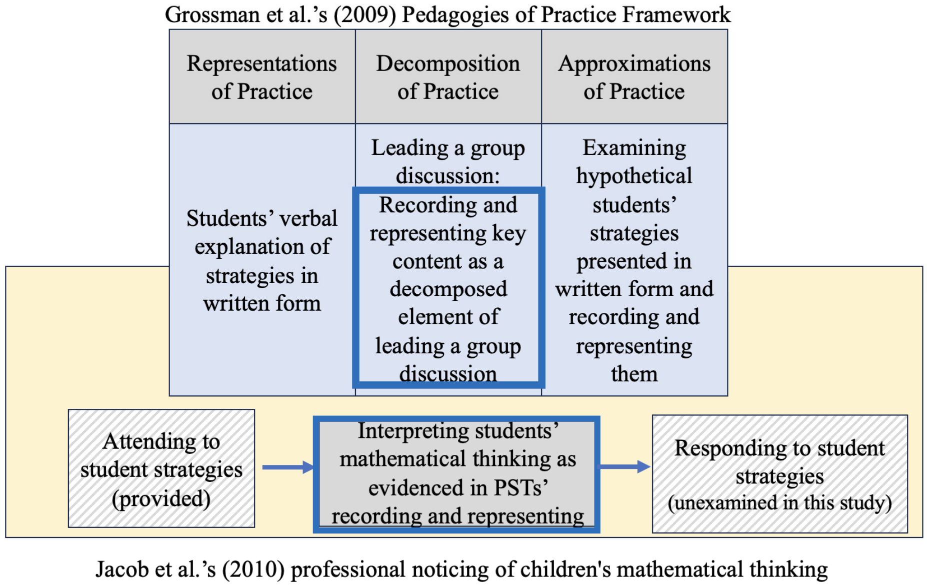
2. Focusing on the "Interpreting" component of the noticing framework

The capacity of teachers to interpret mathematical thinking of students is crucial for ensuring high-quality teaching and learning (NCTM, 2014; TeachingWorks, 2024). Ball et al. (2008) claimed that mathematics teachers need to have the capacity to interpret student mathematical thinking presented in a classroom discussion or written artifacts because it "is an essential part of effectively engaging students in the learning mathematics" (p. 404). This capacity entails the ability to identify key aspects of mathematical thinking, such as strategies that students use to solve a problem and mathematical understanding reflected in their verbal and written works (Jacobs et al., 2010; Sherin et al., 2011; van Es & Sherin, 2002).

Previous studies have pointed out that PSTs face difficulties in identifying and interpreting student strategies (Jacobs et al., 2010; Shaughnessy et al., 2021; Sleep & Boerst, 2012). Sleep and Boerst (2012) asked PSTs to interview students and interpret their mathematical understanding based on the evidence. The authors reported that many PSTs could not properly interpret mathematical thinking of students. PSTs' assertion was too broad, contradicted with the evidence, and lack of clarity. Baldinger (2020) asked PSTs to solve mathematical tasks and then asked them to analyze students' written work who solved the same tasks. Baldinger found that 30% of PSTs' interpretation of the mathematical thinking of students was not aligned with the evidence. The consistent evidence from these articles, indicating that PSTs struggled to interpret mathematical thinking of students, led us to conjecture that they may also face similar challenges in representing and recording student mathematical thinking.

In sum, as shown in Figure 1, this study is situated in the *decomposition of practice* of pedagogies of practice framework (Grossman et al., 2009) and *interpreting students' mathematical thinking* in the noticing framework (Jacobs et al., 2010).

Figure 1 *Conceptual framework of this study*



IV. METHODS

1. Participants and context

This study comprised a total of 90 PSTs, including 45 PSTs enrolled in an elementary mathematics methods course at a university in the Midwestern U.S. and 45 PSTs enrolled in an elementary mathematics course in the Northeastern part of South Korea. All participants were undergraduate elementary education majors pursuing initial teacher certifications. As the study was conducted at two different sites, the experiences of research participants in their respective teacher education programs differed. The participants in the U.S. were in their junior year, while the participants in Korea were juniors (29 PSTs) and seniors (16 PSTs). For both sites, PSTs were required to have field experience hours at local classrooms.

The U.S. elementary teacher education program has recently been revised to incorporate core practices across the program in accordance with the state requirements[3]. This study commenced after the PSTs in the U.S. completed the first mathematics methods class in a series of three, which focused on topics relevant to grades PK-3, which included attribution, counting, and whole number representation. The core practices emphasized in the course was eliciting and interpreting individual student thinking. Activities such as *Which One Doesn't Belong* (see some examples from https://wodb.ca/) were utilized to support this practice. Additionally, each PST was assigned to interview a student in grades PK-1, focusing on their comprehension of counting and problem-solving skills. Data collection for this study occurred at the onset of the second mathematics methods course, which centered on whole numbers and operations for grades 3 - 6, along with core practices.

The PSTs at the Korean site in this study had experience taking at least two mathematics education courses. The first course focuses on the theory of mathematics education, such as mathematics curriculum, the psychology of mathematics teaching and learning, and the teaching methods for mathematics. The second course focuses on mathematics teaching practice in the elementary school environment (grades 1 - 6). In this course, PSTs learned Korea's 2022 revised mathematics curriculum and relevant core practices across mathematical domains (e.g., number and operation, pattern and relation, geometry and measurement, data and chance). They completed at least a four-week-long teaching practicum. During the practicum, they observed mathematics instruction of mentoring teachers and led a mathematics lesson for a whole class. Thus, they had opportunities to elicit and interpret student thinking, lead a group discussion, and explain mathematical contents, ideas, and strategies.

2. Task design and data collection

This study invited PSTs to engage in a task during their regular coursework activities. For the U.S. site, the first author was the course instructor and offered a session introducing the brief classroom conversational routine called Number Talks. The first author also visited the Korean site and offered the same session as a guest instructor to PSTs and collected data from them. A number talk is a brief classroom routine "where students mentally solve computation problems and talk about their strategies" (Humphreys & Parker, 2015, p. 5). It is considered a manageable practice where PSTs learn a variety of important skills, such as posing tasks, asking questions, eliciting student thinking, and recording student strategies for class discussions. Usually, students put paper and pencils away during the number talk because students

3) The list of state's Core Teaching Practices is based on the 19 High-Leverage Practices developed by TeachingWorks (2024).

tend to use rote methods when the writing tools are available. Typically, teachers record student thinking on the board during the number talk. It means that teachers must consider how to "capture each student's thinking so that others may access the strategy" (Parrish, 2014, p.22). As an exploratory study, we collected the data for this study before offering the introductory session of Number Talks through a pre-activity at both research sites in order to uncover PSTs' initial thoughts and performance regarding the teacher's work of recording and representing student thinking.

Participants were given a three-part written task to record and represent four hypothetical strategies that students would use in solving a two-digit addition problem with regrouping (46 + 38). These strategies are likely to arise while eliciting and interpreting student thinking and leading a group discussion in the context of a Number Talk (Humphreys & Parker, 2015). Table 3 presents the four hypothetical students' verbal explanations we provided to PSTs. The table also denotes how we anticipated PSTs would record and explain each student's strategy by listing the key aspects of credible recording and explanations.

Table 3 *Four hypothetical students' strategies for solving 46 + 38 (Adopted from van de Walle et al., 2018)*

Hypothetical students' verbal explanations provided to PSTs	Anticipated key aspects in PSTs' recordings and explanations of the student strategy
Strategy A: 40 and 30 equals 70. 6 and 8 equals 14. 70 and 14 equals 84.	• Decompose each addend by place value (tens and ones) • Add tens and ones • Combine two partial sums
Strategy B: 46 and 30 more equals 76. Then I added on the other 8. 76 and 4 equals 80 and 4 equals 84.	• Decompose the second addend by place value • Starting from the first addend (48), add on tens from the second addend. • When adding ones (8), the student intentionally decomposed 8 into 4 and 4 to make a more manageable number (80) to compute.
Strategy C: Take 2 from the 46 and put it with 38 to equal 40. Now you have 44 and 40 more equals 84.	Strategy C could be interpreted in two ways: *Decompose and move some to make a ten* • Decompose the first addend so that the other addend can be a multiple of 10 for each computation. Student C might apply the associative property of addition: 46 + 38 = (44 + 2) + 38 = 44 + (2 + 38) = 44 + 40 = 84 or *Compensation:* • Subtract a certain quantity from one addend and add the same quantity to the other quantity: 46 + 38 = (46 − 2) + (38 + 2) = 44 + 40 = 84
Strategy D: 46 and 40 equals 86. That's 2 extra, so it's 84.	• Change the second addend to a multiple of 10 for easy computation by adding 2 to 38. • Add the two (modified) addends. • Subtract 2 to compensate.

The four hypothetical student strategies were created based on van de Walle et al.'s (2018) presentation of four different strategies that children use when given addition with two-digit addends. The four strategies allow us to reveal PSTs' initial thoughts and performance of recording and representing student mathematical thinking for three reasons. First, the presented mathematics problem (46 + 38) and student strategies are versatile in terms of their approach to solving the problem and the level of detail in the verbal explanations.

For example, Strategy C explicitly addresses how the initial addends are changed for more manageable computations, while the other strategies have this process implicit. Second, the burden of understanding and solving the mathematical problem (46 + 38) is minimal on the part of PSTs, so they can focus on students' strategies that are asked to be recorded and represented. Third, as the initial learning cycle, the less authentic context (i.e., examining hypothetical student thinking without interactions with the real students in the authentic classroom) could lessen complexity for the participants with limited teaching experiences.

We presented the four strategies on the left side of Table 3 to PSTs and asked them to consider the context in which a short discussion on different strategies took place (e.g., Number Talks). Also, they were asked to interpret and hypothesize students' thought processes based on their verbal explanations to present the most self-explanatory recordings, acknowledging that students might not always offer detailed verbal descriptions (see Table 4).

Table 4 *Prompts presented to PSTs*

Direction: Consider a situation where you facilitate a discussion about various computation strategies for solving 46 + 38 with a group of students. Four students describe their strategies verbally, as follows:

- Strategy A: 40 and 30 equals 70. 6 and 8 equals 14. 70 and 14 equals 84.
- Strategy B: 46 and 30 more equals 76. Then, I added on the other 8. 76 and 4 equals 80 and 4 equals 84.
- Strategy C: Take 2 from the 46 and put it with 38 to equal 40. Now you have 44 and 40 more equals 84.
- Strategy D: 46 and 40 equals 86. That's 2 extra, so it's 84.

For each student's strategy, please present the following in the corresponding section on the provided sheets:

Part 1: How do you want to record and represent each student's explanation on the board so that everyone in the class understands it? Provide your recording of the student's strategy.

Part 2: Briefly explain the key aspects of your record and representation, considering:
(a) How do you decide what needs to get onto the board?
(b) Why do you organize it in a specific way?
(c) Do you think you record/represent so that all students can understand each student's mathematical idea?

Give a short name for the student's strategy.

Part 1 aimed to explore how PSTs document and portray the strategies of individual students on the board for class-wide communication. Parts 2 and 3 aimed to assess whether PSTs accurately interpreted each strategy, particularly in cases where detailed work was lacking in Part 1. Parts 2 and 3 also allow us to examine whether PSTs demonstrated their proficiency in distinguishing and specifying the distinctive features of each strategy while ensuring coherence across all three parts.

3. Data analysis

To answer the first research question of how PSTs record and represent four hypothetical students' strategies, we initially adapted key elements of the skillful use of recordings and representations presented in the decomposition documents of core practices, such as leading a group discussion (TeachingWorks, 2024) as well as studies that address these key elements and principles in the context of mathematics teaching (e.g., Garcia et al., 2021; Shaughnessy et al., 2021). However, this study collected data from the context at the less authentic end of the continuum of approximations of practice, intending to support PSTs in engagement in deliberate practice (Lampert et al., 2013). Therefore, we have made several adjustments to our analytical framework to account for this level of approximation (refer to Table 2 for the development process of the analytical framework). Table 5 provides the description of each aspect that was considered as evidence of key elements of recording and representation in five categories: (a) authenticity, (b) accuracy, (c) organization, (d) support for understanding, and (e) specification. These categories were formulated by drawing upon the key elements highlighted in the earlier studies concerning recording and representing student mathematical thinking (Garcia et al., 2021; Shaughnessy et al., 2021; TeachingWorks, 2024). Each PST's work for the four hypothetical strategies was coded into two categories: (a) evidence is present, or (b) evidence is lacking.

Utilizing the analytical framework outlined in Table 5, two coders independently examined the work of 15 PSTs across the four strategies, representing approximately 17% of the dataset, to assess intercoder reliability. The coders identified the presence or absence of evidence for each subcategory, resulting in an intercoder reliability of 92%. In instances where discrepancies arose in the interpretation of items, a discussion ensued until consensus was achieved. Subsequently, the two coders collaboratively coded the remaining data, resolving discrepancies as they progressed.

Table 5 *Analytical framework*

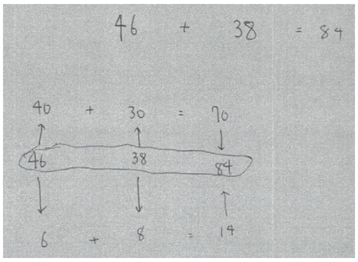
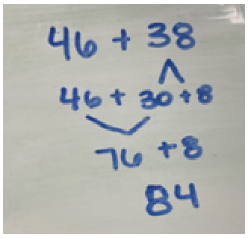
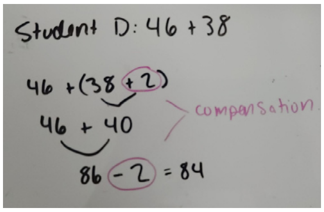
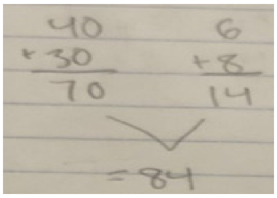
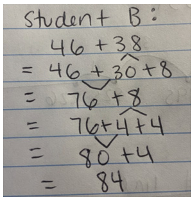
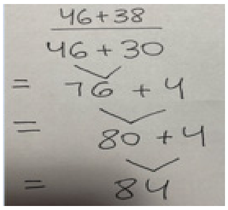
Category	Descriptions and Examples	
A. Authenticity		
A1. Translation	Evidence is present: Translate student verbal explanation into a written/drawn form of record in ways that are true to the student's contribution  46 + 38 = 84 40 + 30 = 70 46 38 84 6 + 8 = 14 Strategy A (KOR#1): The student's contribution of decomposing each addend by its place value and recomposing them by place value is shown without adding additional ideas or detracting from the student's original idea	Evidence is lacking: Key parts are missing, or PSTs add ideas that the student does not say  46 + 38 46 + 30 + 8 76 + 8 84 Strategy B (U.S. #5): Does not show that 8 is decomposed into 4 and 4

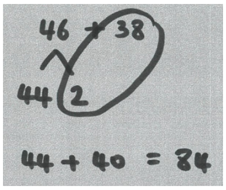
Table 5 *Continued*

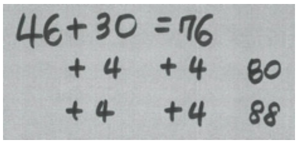
<table>
<tr><th>Category</th><th colspan="2">Descriptions and Examples</th></tr>
<tr><td>A2. Thinking paths</td><td>Evidence is present: Delineate student thinking paths


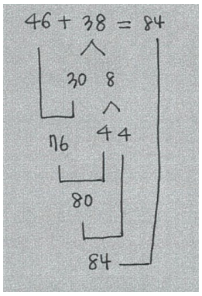
Strategy D (U.S. #1): Explicitly shows the student's thinking path of initially reaching "40" and subsequently subtracting "2" at the conclusion to compensate</td><td>Evidence is lacking: Recording only shows word-for-word (or sentence-by-sentence) translation without revealing the full thinking paths


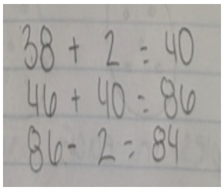
Strategy A (U.S. #8): Does not explicitly show how each addend is decomposed by place value</td></tr>
<tr><td>B. Accuracy</td><td></td><td></td></tr>
<tr><td>B1. Notation</td><td>Evidence is present: Attend to accuracy in mathematical notations


Strategy B (U.S. #12): The equal sign is appropriately used as a symbol denoting the equivalent relationship</td><td>Evidence is lacking: Incorrect use of mathematical notations


Strategy B (U.S. #38): Incorrect use of the equal sign</td></tr>
<tr><td>B2. Value</td><td>Evidence is present: Attend to accuracy in mathematical computations


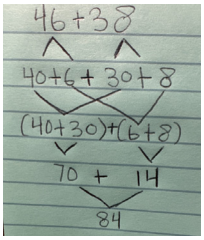
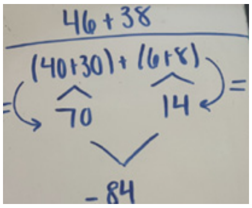
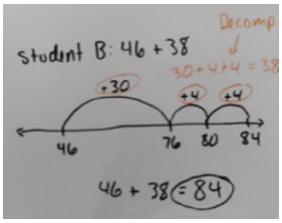
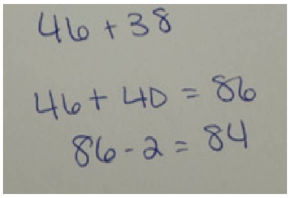
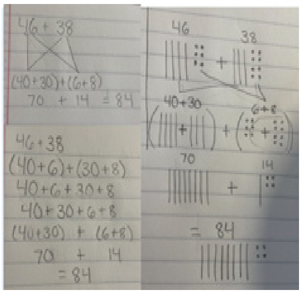
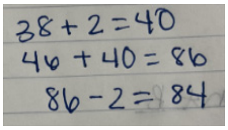
Strategy C (KOR#6): Correct computations</td><td>Evidence is lacking: Inaccurate mathematical computations


Strategy B (KOR#9): Incorrect final answer</td></tr>
<tr><td>C. Organization</td><td></td><td></td></tr>
<tr><td>C1. Clarity and flow</td><td>Evidence is present: Create self-explanatory recordings with clarity and flow.


Strategy B (KOR#23): Explicitly shows the student's thinking process in a clear and sequential manner</td><td>Evidence is lacking: Recording is not self-explanatory. By looking at the recording, it cannot be determined what problem is being solved.


Strategy D (U.S. #25): The original problem is not recorded, and it is not clear where each equation comes from</td></tr>
</table>

Table 5 *Continued*

Category	Descriptions and Examples	
C2. Consistency (notation)	Evidence is present: Use representational symbols and notations in a consistent manner  Strategy A (U.S. #7): Consistent use of schematics for composing/decomposing processes	Evidence is lacking: Use the same symbol or notation for different processes (inconsistency)  Strategy A (U.S. #32): Inconsistent use of schematics for composing numbers
D. Support for Understanding		
D1. Added details	Evidence is present: Add intentional annotations and markers to support sense-making that go with numeric expressions (e.g., parentheses, annotations in words, schematics, arrows, circles, colors, etc.)  Strategy B (U.S. #1): Annotated how the second added is decomposed and showed how each part is sequentially added using the number line and used different colors	Evidence is lacking: No additional markers in addition to numeric-only recording  Strategy D (U.S. #19): No effort to add details to highlight the key ideas other than numeric expressions
D2. Visual representations	Evidence is present: Utilize visual representations other than numeric-only representation along with or without added details/markers  Strategy A (U.S. #27): equations, schematics, and drawn base 10 blocks were used	Evidence is lacking: No visual representations are used beyond numeric-only recording  Strategy D (U.S. #20): No visual representations were used other than symbolic-only recordings

 https://doi.org/10.29275/jerm.2024.34.3.511

Table 5 *Continued*

Category	Descriptions and Examples	
E. Specification		
E1. Explanation (interpretation, Part 2 of Table 4)	Evidence is present: Explain/Interpret the student's core ideas with specification Strategy A (U.S. #20): *"From the student's explanation, I can see that they* decomposed both addends by place value *to get 40+30 and 6+8. So, to clarify to the other students how the student got his answer, I recorded this way..."* [This explanation noted a specific decomposition method: decompose by place value.]	Evidence is lacking: Provide generic statements without specification or misinterpret the student's core ideas Strategy A (U.S. #15): *"I organized the information going down to show each step as it progresses. I used lines to show how the numbers were broken up and parentheses to show those 2 numbers make up the number that was broken up."* [This explanation does not indicate a specific decomposition method.]
E2. Naming (Part 3 of Table 4)	Evidence is present: Name the strategy in a way that highlights the most critical aspects of the student's strategy with specification Strategies A and B (U.S. #12): In Part 3, Student A's strategy is named "decomposition by place value for addition," and Student B's strategy is referred to as "decomposition to make friendly numbers."	Evidence is lacking: Names the strategy very generally without specifications Strategies A and B (U.S. #7): In Part 3, both strategies are named "decomposing numbers" without specifics on different methods of decomposing numbers.

V. RESULTS

1. Overview of all PSTs' work

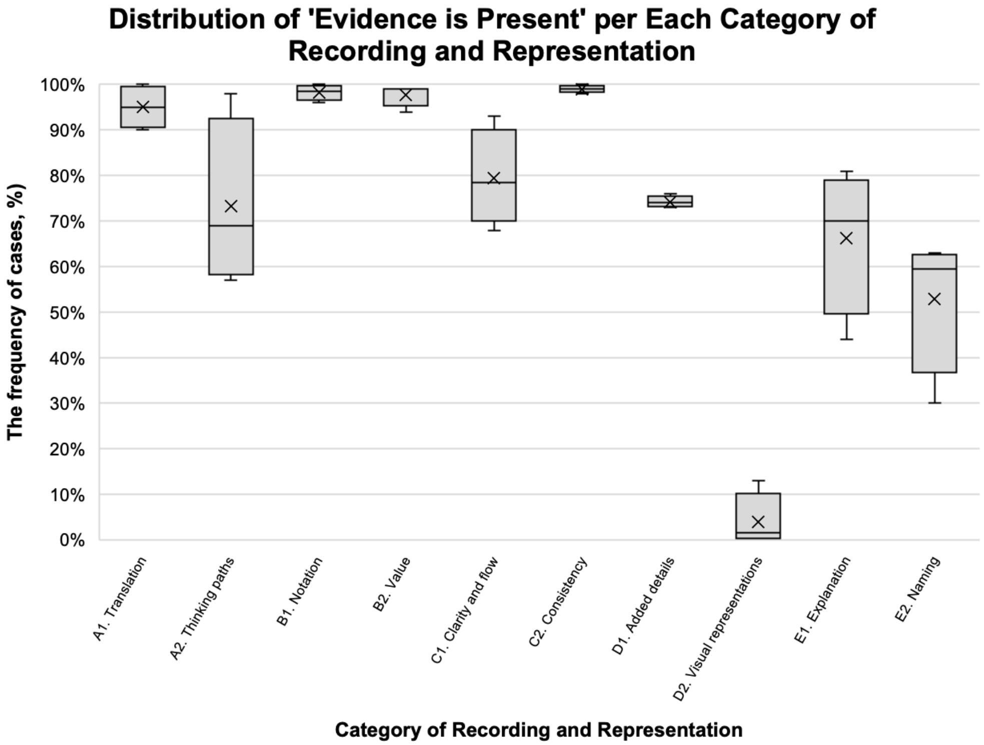
In addressing the first research question, we scrutinized all PSTs' recordings and representations in Part 1, explanations in Part 2, and the labels of the four hypothetical students' strategies noted in Part 3 (see Table 4). This analysis involved determining the presence or absence of evidence related to the principles and elements outlined in the analytical framework (see Table 5). PSTs' performances were classified into "Outperformed," "Moderately performed," and "Underperformed." "Outperformed," "Moderately performed," and "Underperformed" indicate the category where "more than 90%", "between 70%-80%", and "less than 10%" of PSTs show evidence in their response. "Varying performed" indicates a category showing that PSTs' evidence fluctuates across strategies. Across the four strategies, PSTs showed different levels of performance, while the differences were not large, from 69% (Strategy B) to 81% (Strategy C). A boxplot (Figure 2) illustrates the distribution of frequencies of cases (%) categorized as 'evidence is present' for each subcategory and according to the given student strategy. Additionally, Table 6 provides a summary of the findings across all participants.

1) Outperformed categories: A1, B1, B2, and C2

As shown in Table 6 and Figure 2, categories exhibiting high frequencies of instances where 'evidence is present' across all four student strategies include A1 (translation, range = 90-100%, mean (S.D.) = 95%

(4%)), B1 (notation, range = 96-100%, mean (S.D.) = 98% (2%)), B2 (value, range = 94-99%, mean = 98% (2%)), and C2 (consistency, range = 98-100%, mean = 99% (1%)). This finding implies that PSTs are likely to be able to translate students' verbal explanations into a written/drawn form of record (A1). In addition, they tend to use accurate mathematical notations (B1) and mathematical values (B2). Also, PSTs tend to use presentational symbols and notations in a consistent manner (C2).

Figure 2 *Boxplot that shows the distribution of the frequency of cases categorized as 'evidence is present for each category of recording and representation across four strategies.*

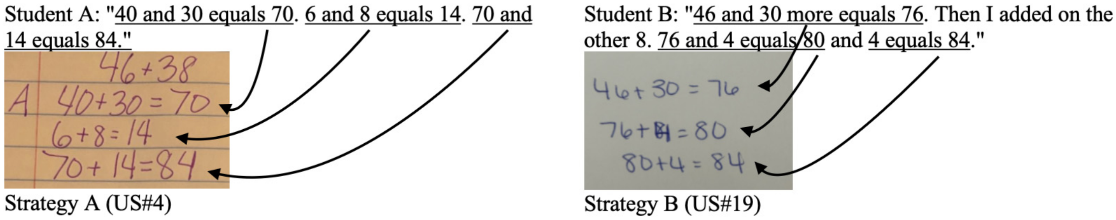


Note. 'X' in the box indicates the mean value. Based on the frequency of cases that 'evidence is present,' PSTs' performances were classified into underperformed (D2), moderately performed (D1), outperformed (A1, B1, B2, and C2), and varying performance (A2, C2, E1, and E2). "Outperformed," "Moderately performed," and "Underperformed" indicate the category where "more than 90%", "between 70%–80%", and "less than 10%" of PSTs show evidence in their response. "Varying performed" indicates a category showing that PSTs' evidence fluctuates across strategies.

Within A1 (authenticity), the predominant code for PSTs' work was 'evidence is present' (range 90-100%). A significant portion of the PSTs' work was identified as direct word-for-word translations of each student's strategy, which means PSTs recorded students' explanations without altering their contributions (see Figure 3). Note that although these two instances (Figure 3) are classified as 'evidence is present' for A1 (authenticity), these two do not clearly present student thinking paths (A2) or add details to support student understanding

(D1). This result indicates that performance in recording and representing student mathematical thinking varies depending on the category.

Figure 3 *Examples of PSTs' responses related to category A1 (authenticity)*



Regarding B1 (notation), six out of a total of 360 cases (2%) displayed incorrect usage of mathematical notations, all of which involved misusing the equal sign. For example, one PST in the U.S. (ID #38) represented that 76 + 4 = 80 + 4. For category B2 (value), seven instances (2%) were categorized as 'evidence is lacking' due to inaccurate numerical recordings. We interpreted six of these seven instances (misusing equal signs and inaccurate numerical recording) as likely stemming from careless writing or calculation errors rather than conceptual misunderstandings. For instance, one PST in the U.S. (ID #45) recorded Strategy B by noting 48 as the first addend rather than 46. Despite this initial mistake, the subsequent recording accurately displayed the addition results (see Figure 4).

Figure 4 *Example of inaccurate recording due to careless error*

Apart from these instances, all other cases were classified as 'evidence is present' except for one with no response. This result may be attributable to the fact that the tasks used in this study did not involve a high level of complexity or cognitive demand for PSTs.

C2 (consistency) showed a high frequency of 'evidence is present.' However, it is notable that PSTs generally did not employ a wide range of representational symbols and notations beyond numeric-only number sentences with some annotations and symbols such as + and =. Possible notations that could have been used are informal notations for decomposition [^] and recomposition [v] that PSTs were exposed

Table 6 *Summary of responses from participating PSTs in the U.S. and Korea*

Category		Strategy A			Strategy B			Strategy C			Strategy D			Mean (S.D.) Classification
		Yes	No	NA	Yes	No	NA	Yes	No	NA	Yes	No	NA	
A. Authenticity	A1. Translation	90 (100%)	0 (0%)	0 (0%)	81 (90%)	9 (10%)	0 (0%)	88 (98%)	2 (2%)	0 (0%)	83 (92%)	6 (7%)	1 (1%)	95% (4%) Outperformed
	A2. Thinking paths	51 (57%)	39 (43%)	0 (0%)	68 (76%)	22 (24%)	0 (0%)	88 (98%)	2 (2%)	0 (0%)	56 (62%)	33 (37%)	1 (1%)	73% (16%) Varying performed
B. Accuracy	B1. Notation	90 (100%)	0 (0%)	0 (0%)	88 (98%)	2 (2%)	0 (0%)	89 (99%)	1 (1%)	0 (0%)	86 (96%)	3 (3%)	1 (1%)	98% (2%) Outperformed
	B2. Value	89 (99%)	1 (1%)	0 (0%)	85 (94%)	5 (6%)	0 (0%)	89 (99%)	1 (1%)	0 (0%)	89 (99%)	0 (0%)	1 (1%)	98% (2%) Outperformed
C. Organization	C1. Clarity and Flow	68 (76%)	22 (24%)	0 (0%)	61 (68%)	29 (32%)	0 (0%)	84 (93%)	6 (7%)	0 (0%)	73 (81%)	16 (18%)	1 (1%)	80% (9%) Varying performed
	C2. Consistency	89 (99%)	1 (1%)	0 (0%)	88 (98%)	2 (2%)	0 (0%)	90 (100%)	0 (0%)	0 (0%)	89 (99%)	0 (0%)	1 (1%)	99% (1%) Outperformed
D. Supporting Understanding	D1. Added details	67 (74%)	23 (26%)	0 (0%)	66 (73%)	24 (27%)	0 (0%)	68 (76%)	22 (24%)	0 (0%)	67 (74%)	22 (24%)	1 (1%)	74% (1%) Moderately performed
	D2. Visual representations	2 (2%)	88 (98%)	0 (0%)	12 (13%)	78 (87%)	0 (0%)	0 (0%)	90 (100%)	0 (0%)	1 (1%)	88 (98%)	1 (1%)	4% (5%) Underperformed
E. Specification	E1. Explanation	60 (67%)	30 (33%)	0 (0%)	40 (44%)	48 (53%)	2 (2%)	73 (81%)	13 (14%)	4 (4%)	66 (73%)	16 (18%)	8 (9%)	66% (14%) Varying performed
	E2. Naming	51 (57%)	32 (36%)	7 (8%)	27 (30%)	59 (66%)	4 (4%)	56 (62%)	26 (29%)	8 (9%)	57 (63%)	17 (19%)	16 (18%)	53% (14%) Varying performed
Mean		66 (73%)	24 (27%)	0 (0%)	62 (69%)	28 (31%)	1 (1%)	73 (81%)	16 (18%)	1 (1%)	66 (74%)	21 (23%)	3 (3%)	

Note. "Yes" or "No" indicates that evidence for such a category is present or lacking in the PST response. "N.A." denotes the instances in which PSTs did not provide their responses. "Outperformed," "Moderately performed," and "Underperformed" indicate the category where "more than 90%", "between 70%–80%", and "less than 10%" of PSTs show evidence in their response. "Varying performed" indicates a category showing that PSTs' evidence fluctuates across strategies.

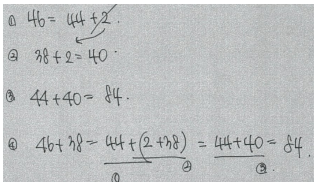
to during the course and their field experiences. Consequently, PSTs' responses did not exhibit inconsistent use of symbols or notations due to the limited variety used, resulting in a high frequency of 'evidence is present' in C2.

Overall, although PSTs excelled in several categories (A1, B1, B2, and C2), the high face values may not inherently signify PSTs' adeptness in recording and representation.

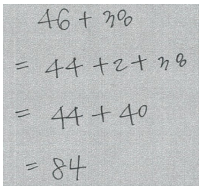
2) Moderately performed categories: D1

In D1 (added detail), PSTs displayed moderate usage of incorporating additional details to enhance student comprehension (range = 73-76%), mean (S.D.) = 74% (1%)). These supplementary details encompassed various elements such as words, arrows, schematics (e.g., decomposition [^] and recomposition [v]), utilization of distinct colors, and annotations to emphasize key concepts and procedures. Mere circling or underlining to denote final numerical answers was not deemed sufficient evidence to provide additional details for supporting student comprehension. Instances categorized as 'evidence is lacking' primarily involved recording solely numerical sentences (see Figure 5, right figure).

Figure 5 *Examples of PSTs' responses related to category D1 (Added details)*



Evidence is present (KOR#28): Displayed every step of the process along with annotations indicating the sequence of each step within the overall process

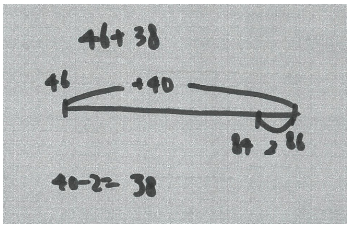


Evidence is lacking (KOR#15): No effort to add details to highlight the key ideas other than numeric expressions

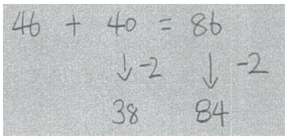
3) Underperformed categories: D2

The category with the lowest number of PSTs showing evidence was *visual representations* (D2). Visual representations were utilized in only 15 cases across four strategies (range 0-13%, mean (S.D.) = 4% (5%)). 14 of them used number line representation (see Figure 6), while the other one instance involved base ten block representations. Importantly, 12 out of the 15 cases were observed in the recording and representation of Student B's strategy ("46 and 30 more equals 76. Then, I added on the other 8. 76 and 4 equals 80, and then 80 and 4 equals 84."). The remaining recordings were represented with only symbols (e.g., equations). Furthermore, it was rare to find recordings that attempted to present multiple representations with symbols and visual models.

Figure 6 *Examples of PSTs' responses related to category D2 (visual representations)*



Evidence is present (KOR#13): A number line is used along with number sentences.



Evidence is lacking (KOR#22): No visual representations were used other than symbolic-only recordings along with annotations

4) Varying performed categories: A2, C1, E1, and E2

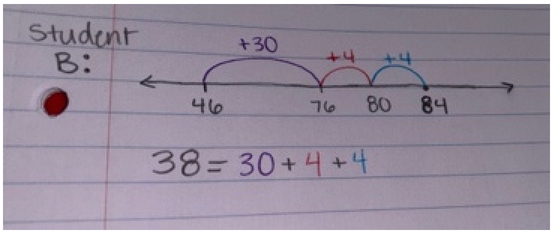
A2 (thinking paths), C1 (clarity and flow), E1 (explanation), and E2 (naming), coded as 'evidence is present,' exhibited a high standard deviation (> 6%), meaning a broad spectrum of frequencies depending on the presented student strategy. For example, In A2, 98% of PSTs' recording and representation of Strategy C was coded as 'evidence is present,' followed by Strategy B (76%), Strategy D (62%), and Strategy A (57%).

Additionally, Strategy A demonstrated the fewest instances labeled as 'evidence is present' despite being the common addition strategy associated with the standard algorithm emphasizing addition by place value (Strategy A: "40 and 30 equals 70. 6 and 8 equals 14. 70 and 14 equals 84.") Many PSTs did not indicate how each addend was broken down by place value when employing Strategy A. We conjectured that PSTs might not provide thorough descriptions of student thinking pathways under the assumption that these pathways are apparent without explicit records and representations. Likewise, Strategy D ("46 and 40 equals 86. That's 2 extra, so it's 84.") was another one where many PSTs did not explicitly record the thought process behind choosing to add '40' instead of 38. Rather, they simply recorded "46 + 40 = 86" and "86 - 2 = 84."

Remarkably, Strategy C ("Take 2 from the 46 and put it with 38 to equal 40. Now you have 44 and 40 more equals 84.") stood out as the only strategy where the student verbatim explicitly referred to both addends (46 and 38), in contrast to the other strategies (A, B, and D). This finding implies that when both addends are explicitly mentioned, PSTs are likely to delineate student thinking paths more clearly.

In E1 (explanation), when PSTs provided a generic statement without addressing key ideas or intention behind the strategies, simply repeating the presented student's verbatim, and misinterpreting strategies, we coded them as 'evidence is lacking.' Strategy B ("46 and 30 more equals 76. Then, I added on the other 8. 76 and 4 equals 80, and then 80 and 4 equals 84.") showed a distinctively low frequency of cases coded as 'evidence is lacking' for the abovementioned reasons. For example, in Figure 7, this PST (U.S. #17) explained that Student B employed the "doubles" strategy. However, the PST did not explicitly explain the meaning of double and what numbers are doubled (e.g., 8 was decomposed into two 4s).

Figure 7 *The response from one PST (U.S. #17) for Strategy B*



In E2 (naming), 47% of the PSTs' responses were coded as 'evidence is lacking.' Many of them were due to the utilization of broad labels without specific details. For example, a PST in the U.S. (ID #1) named Strategies A and B as "decomposition." Although decomposition was indeed present across various strategies, this generic label failed to differentiate the distinctive features of each strategy. For example, as previously mentioned, most visual representations were associated with recording and representing Strategy B, with a significant portion being number lines. Thus, some PSTs named the strategy by highlighting the model they utilized (e.g., "number line strategy") instead of the key aspect of the strategy.

2. Comparison

We examine the differences in performance of PSTs in the U.S. and Korea, concentrating on distinctive aspects of their responses by category related to recording and representing student mathematical thinking. Table 7 shows that across the four strategies, participating PSTs in the U.S. outperformed their counterparts in Korea regarding the presence of evidence. Moreover, except for B1 and C2, PSTs in the U.S. showed better performances than PSTs in Korea in the other eight categories; the differences between the two groups in B1 (3%) and C2 (1%) are negligible. These findings indicate that overall, the participants at the U.S. site showed better performances than the PSTs at the Korean site (80% and 69% of evidence is present, respectively) in recording and representing student mathematical thinking in the addition of two-digit numbers. Although the PSTs at the Korean site had lower scores than their counterparts, their scores for Strategy 1 and Strategy 2 were slightly higher in Categories C2, B1, B2, and E1. We grouped these findings into (1) minor discrepancies (A1, B1, B2, C2, D2, and E1), where the difference between the two groups is less than or equal to 10% and (2) moderate to major discrepancies (A2, C1, D1, and E2), where the difference between the two groups exceeds this threshold (see Table 7).

Table 7 *Comparison of the performances of PSTs in the U.S. and Korea focusing on 'yes'*

Category	Strategy A		Strategy B		Strategy C		Strategy D		Mean Value Classification	
	U.S.	K.O.	U.S.	K.O.	U.S.	K.O.	U.S.	K.O.	U.S.	K.O.
A1. Translation	100%	100%	89%	91%	98%	98%	98%	87%	96%	94%
									Minor Discrepancy	
A2. Thinking paths	71%	42%	82%	69%	98%	98%	87%	38%	85%	62%
									Moderate to Major Discrepancy	
B1. Notation	100%	100%	96%	100%	98%	100%	93%	98%	97%	100%
									Minor Discrepancy	
B2. Value	100%	98%	93%	96%	98%	100%	100%	98%	98%	98%
									Minor Discrepancy	
C1. Clarity and Flow	96%	56%	82%	53%	100%	87%	96%	67%	94%	66%
									Moderate to Major Discrepancy	
C2. Consistency	98%	100%	96%	100%	100%	100%	100%	98%	99%	100%
									Minor Discrepancy	
D1. Added details	84%	64%	93%	53%	76%	76%	73%	76%	82%	67%
									Moderate to Major Discrepancy	
D2. Visual representations	4%	0%	27%	0%	0%	0%	0%	2%	8%	1%
									Minor Discrepancy	
E1. Explanation	62%	71%	49%	40%	87%	76%	84%	62%	71%	62%
									Minor Discrepancy	
E2. Naming	58%	56%	36%	24%	80%	44%	89%	38%	66%	41%
									Moderate to Major Discrepancy	
Mean	77%	69%	74%	63%	84%	78%	82%	66%	80%	69%

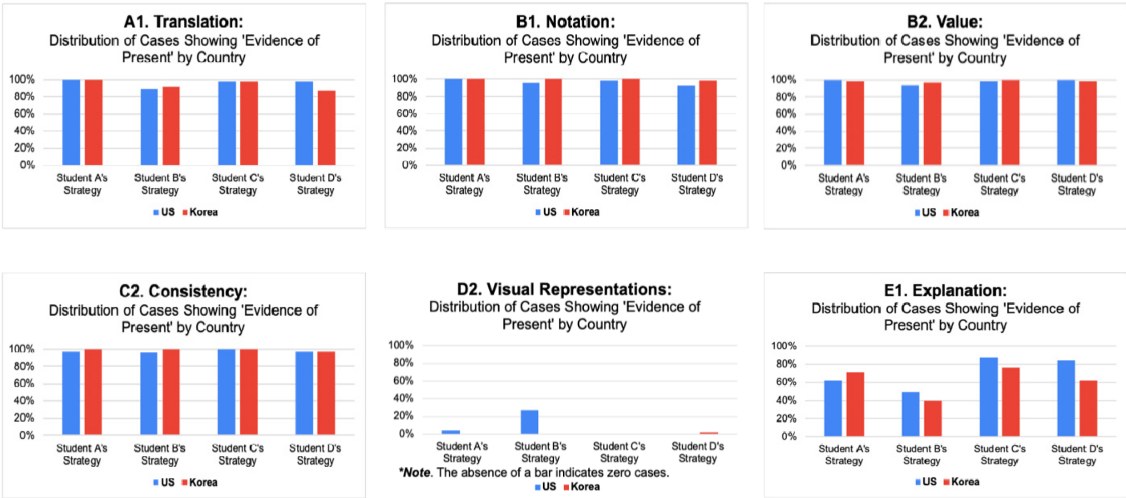
Note. "Yes" indicates that evidence for such a category is present in the PST response. " Minor Discrepancy " indicates that the difference between the two groups on the "Yes" was less than 10%, and "Moderate to Major discrepancies" indicates that the difference between the two groups exceeds this threshold.

1) Minor discrepancies

Regarding the five categories showed outperformed (A1, B1, B2, and C2) and underperformed (D2), the disparities between the two groups in the U.S. and Korea are minor (see Figure 8), ranging from 0% (B2) to 7% (E1). While E1 showed varying performance, the disparities between PSTs are minor (9%). Several noteworthy aspects were observed.

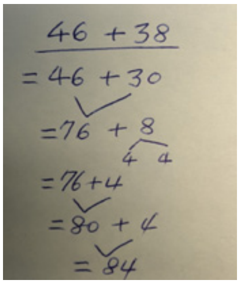
In A1 (translation), both PST groups were able to translate student explanations into a written or drawn form of record at a high degree of accuracy (96% and 94%, respectively). In comparison to Strategies A, C, and D, translating Strategy B showed relatively low frequencies of evidence in both groups (PSTs in the U.S.: 89%, PSTs in Korea: 91%). The primary reason was the failure to record and represent the process of decomposing 8 into 4 and 4.

Figure 8 *Categories that showed minor discrepancies in 'evidence is present' between PSTs in the U.S. and Korea*

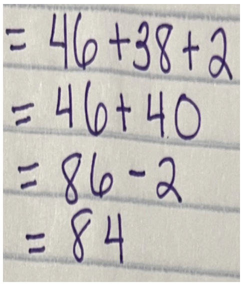


In B1 (notation), PSTs in both the U.S. and Korea showed an accurate use of notation (97% and 100%, respectively). Only six instances of incorrect usage (2%) were found in U.S. PSTs' work where they misused the equal sign (see Figure 9). Other than this, our analysis showed that both groups were able to use correct notations.

Figure 9 *Examples of B1 showing incorrect usage of notation*



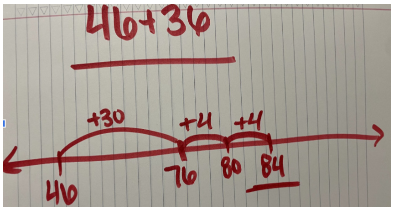
Strategy B (U.S. #21)



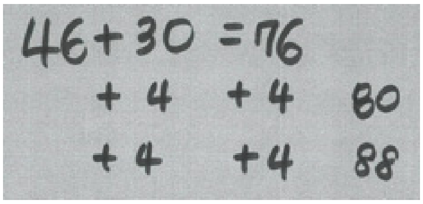
Strategy D (U.S. #38)

In B2 (value), 98% of both groups accurately recorded the numerical values that were expressed in the four strategies. Only four PSTs in the U.S. and three PSTs in Korea showed incorrect numerical values in their responses, which stemmed from a misunderstanding or misinterpretation of the given strategy. Two examples are provided in Figure 10.

Figure 10 *Two examples of B2 showing incorrect usage of values*



Strategy B (U.S. #33)

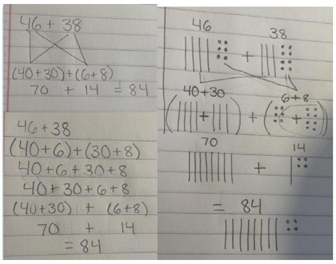


Strategy B (KOR#9)

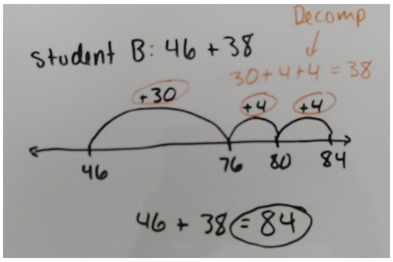
In C2 (consistency), 99% of PSTs in the U.S. and all PSTs in Korea used representational symbols and notations in a consistent manner. Only two U.S. PSTs' cases showed inconsistent use of schematic notations for composing and decomposing numbers.

D2 (visual representations) was a unique category where the least number of cases were coded as 'evidence is present' across both groups of PSTs (PSTs in the U.S.: 8%, PSTs in Korea: 1%). This implies that both groups barely utilized visual representations other than numeric-only representation to record the mathematical ideas presented in the four strategies. Only 15 instances incorporated visual representations, exclusively number lines, except for one example of using base ten block representation. Out of 15 instances, 14 cases were observed in the U.S. PSTs' work (12 for Strategy B and 2 for Strategy A), and only one PST at the Korean site showed a number line for Strategy D (see Figure 11).

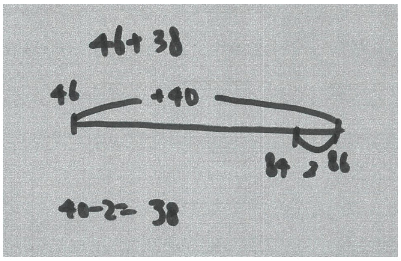
Figure 11 *Examples of D2 showing PSTs' use of visual representations*



Strategy A (U.S. #27)



Strategy B (U.S. #1)



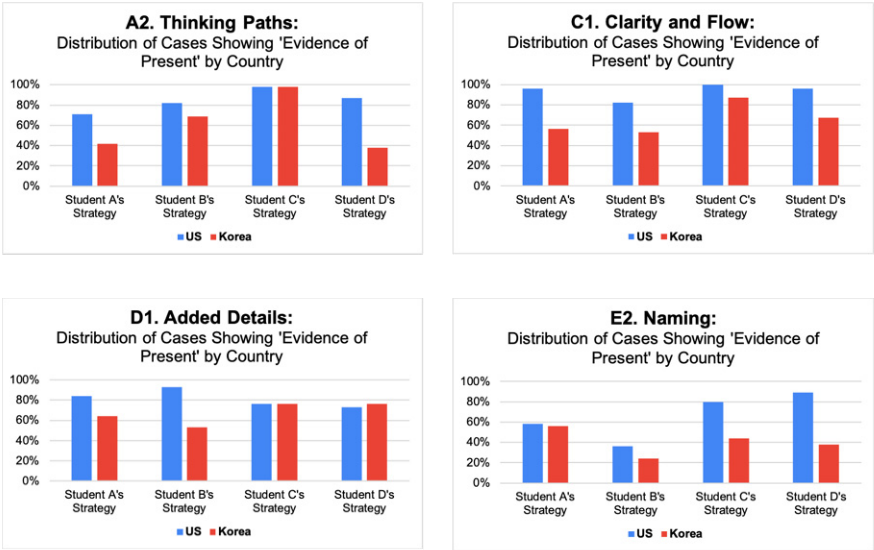
Strategy D (KOR#13)

In E1 (Explanation), 71% of PSTs in the U.S. and 62% of PSTs in Korea explained and interpreted the students' core ideas with specifications. All the cases from both PST groups were coded as 'evidence is lacking' and resulted from unspecified and generic explanations or simply repeating the student verbatim. The exception was found in three instances among the U.S. PSTs where the student reasoning was misinterpreted. This result shows that although both groups can translate student mathematical thinking (A1), their ability to explain and interpret the core ideas is relatively limited.

2) Moderate to major discrepancies

In this section, we examine the categories (A2, C1, D1, and E2) that showed moderate to major discrepancies in 'evidence of present' between PSTs in the U.S. and Korea, ranging from 15% (D1) to 28% (C1). Figure 12 illustrates the overview performance by strategy and by country.

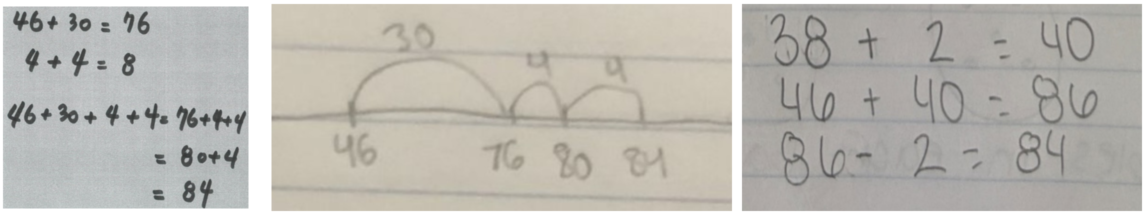
Figure 12 *Categories that showed moderate to major discrepancies in 'evidence is present' between PSTs in the U.S. and Korea*



In A2 (thinking paths), there was a 23% performance gap between the two groups in their ability to delineate student thinking paths (PSTs in the U.S.: 85%, PSTs in Korea: 62%). In both PST groups, most cases coded as "evidence is lacking" were attributed to recordings that were mostly word-for-word translations without showing the sequence of students' thought processes exhibited in the strategy, such as how each addend is decomposed. This tendency was more apparent among PSTs in Korea across the four strategies. This result implies that PSTs in Korea are less likely than U.S. PSTs to delineate student thinking paths as they are.

Category C1 (clarity and flow) delves into whether PSTs created self-explanatory recordings with clarity and flow. Our finding showed a 28% performance gap between PSTs in the U.S. (94%) and their counterparts (66%) in this category. This finding suggests that U.S. PSTs are more likely to accurately record and represent student thinking with clarity and coherence. However, the reasons why their responses were recorded as 'evidence is lacking' were similar. This was due to the original problem with two addends not being recorded or word-for-word translations being used (see Figure 13). Consequently, how the original addends were transformed to facilitate easy calculations was not clearly documented.

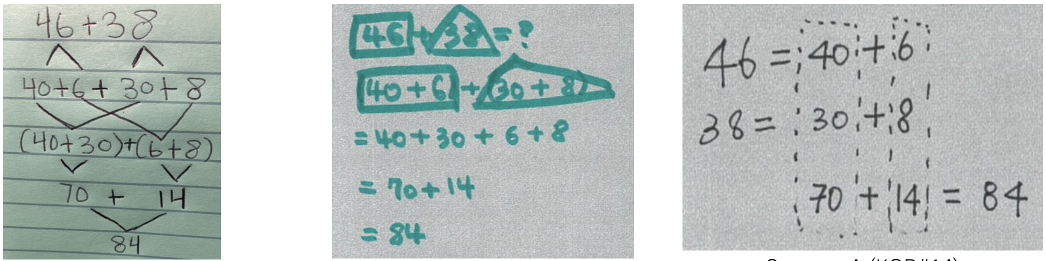
Figure 13 *Examples of C2 showing PSTs' lack of clarity and flow*



Strategy B (KOR#9) Strategy B (U.S. #8) Strategy D (U.S. #25)

In D1 (added details), 82% of PSTs in the U.S. and 67% of PSTs in Korea added intentional annotations and markers that go with numeric expressions to support sense-making. The PSTs in both groups incorporated diverse details and markers to enhance student comprehension. However, U.S. PSTs showed better performance in D1. The details and markers that both PSTs included were words, arrows, schematics (e.g., decomposition [^] and recomposition [v]). They also used different colors and annotations to highlight key concepts and procedures. One notable difference between the two groups is that U.S. PSTs predominantly utilized typical schematic notations to denote the locations of decompositions and recompositions (see U.S. #7 case in Figure 14). In contrast, PSTs in Korea appeared to employ a broader range of methods to add details, as illustrated in the following examples in Figure 14, such as dotted brackets (see KOR #14).

Figure 14 *Examples of D1 showing PSTs' use of notations*



Strategy A (U.S. #7) Strategy A (KOR#4) Strategy A (KOR#14)

Typical notations used by PSTs in the U.S. Various notations among PSTs in Korea

E2 (naming) concerns whether PSTs can name the most critical aspects of students' thinking exhibited in their strategy. Only 64% of U.S. PSTs and 41% of PSTs in Korea showed their ability to do such work. Also, across four students' strategies, U.S. PSTs consistently showed higher performance than PSTs in Korea. One notable aspect is that PSTs in Korea left Part 3 unanswered (9-33%). Looking at the differences in responses between the two groups, PSTs in both the U.S. and Korea lacked explaining a specific decomposition method. For example, there was no indication of whether the decomposition was being done based on place value or not (e.g., "decomposing" versus "decomposing by place value"). Although some PSTs in Korea used various labels (e.g., "step-by-step," "listing numbers," and "intuition"), these labels did not fully represent the essence of the student's mathematical thinking.

VI. DISCUSSION

This study explored the work of PSTs in both the U.S. and Korea in recording and representing four different addition computation strategies. The grain size of this teaching practice is relatively small, yet findings show that this work involves multiple aspects and layers of teacher skills and understanding. Given that this study's data were collected from diagnostic pre-activities before delving into the core teaching practices such as leading a group discussion, the findings of this study can offer insights into the perceptions regarding the work of recording and representing that PSTs brought to the teacher education program and how teacher educators can better support them in this area during their training. In this section, we revisit the study's findings for further discussion.

1. Overall performance

The analytical framework was derived from prior studies and resources related to core practices (e.g., Garcia et al., 2021; Shaughnessy et al., 2021; TeachingWorks, 2024). Regarding the first research question, we found that PSTs showed different performances across the 10 categories: outperformed (A1, B1, B2, and C2), moderately performed (D1), underperformed (D2), and varying performed (A2, C1, E1, and E2). These findings indicate that PSTs tend to accurately translate students' verbal explanations into a written and drawn form of record (A1, translation), record accurate mathematical notations (B1, notation) and mathematical values (B2, values), and use presentational symbols and notations in a consistent manner (C2, consistency). They are also somewhat inclined to incorporate additional details to enhance student comprehension (D1, added details). However, PSTs are not likely to use visuals (D2, visual representations) to represent student mathematical ideas. Considering the existence of varying performed categories, we could conclude that PSTs showed different levels of performance to record and represent the student mathematical thinking represented in the four strategies.

2. Similarities between the PSTs in the U.S. and Korea: Direct translation into numeric-only recordings

We discovered that one significant trend influencing the performance of PSTs in both the U.S. and Korea across five categories (A. Authenticity, B. Accuracy, C. Organization, D. Support for understanding, and E. Specification) was their tendency to directly translate student verbatims into numeric-only recordings. The majority of PSTs in both groups performed word-for-word or sentence-by-sentence translations of student verbatims, despite being instructed to interpret and hypothesize about student thinking processes. Although a transcription of students' verbalizations of their mathematical thinking may be viewed as authentic, it does not fully capture student thought processes. It is evidenced by the relatively low frequencies of 'evidence is present' in Category A2 (thinking paths), contrasted with the frequent occurrences in Category A1 (accurate translation).

Additionally, within Category A2 (thinking paths), Strategy C had a higher frequency of being coded as "evidence is present" compared to the other three strategies (see Table 4). It was attributed to the fact that Strategy C explicitly outlined the transformation of the original addends before their manipulation. In contrast, the other three strategies directly proceeded to manipulate the numbers without explicitly mentioning how the original addends were transformed. The PSTs' approach to the three strategies is not ideal as it provides ambiguous recordings rather than clarifying their interpretation of student mathematical thinking.

Moreover, the relatively low frequency of occurrences of 'evidence is present' in Category C1 (clarify and flow) can be attributed to instances of numeric-only translations provided word-for-word or sentence-by-sentence. Several isolated lines of numeric sentences do not indicate their source or relevance. Therefore, they are not sufficient for self-explanatory recordings. The tendency of presenting direct translation into numeric-only recordings is related to another notable aspect observed in both groups in Category D2 (visual representations).

3. Differences between the PSTs in the U.S. and Korea: U.S. PSTs tend to delineate student thinking pathways and flow in detail

We found some discrepancies between the two PST groups in their practice of recording and representing student mathematical thinking. PSTs in the U.S. performed better than PSTs in Korea in seven out of ten categories (e.g., A1. Translation). While PSTs in Korea outperformed U.S. PSTs in B1 (notation) and C2 (consistency), the differences are negligible (3% and 1%, respectively).

PSTs in the U.S. outperformed PSTs in Korea in delineating student thinking paths (A2. Thinking paths), accurately recording and representing student thinking with clarity and coherence (C1. Clarity and flow), adding details and markers to enhance student comprehension (D1. Added details), and naming the most critical aspects of students thinking exhibited in their strategy (E2. Naming). These findings imply that the PSTs in Korea were more likely to use word-for-word or sentence-by-sentence translation methods without showing the sequence of students' thought processes exhibited in the strategy.

VII. CONCLUSION

Recording and representing student mathematical ideas is an important aspect in leading a group discussion

(Shaughnessy et al., 2021; TeachingWorks, 2024). Our study extended the findings from previous studies (Garcia et al., 2021; Shaughnessy et al., 2021) by demonstrating specific strengths and weaknesses of PSTs in the U.S. and Korea in their skills of recording and representing students' mathematical thinking. Their lack of competencies in some categories can be attributed to several factors, such as their limited teaching experiences and mathematical knowledge, insufficient exposure to the strategies of others, and reliance on their own prior experiences as students (e.g., Lee & Lee, 2023; Namakshi et al., 2022; Wilson et al., 2011). For example, the limited use of visual representations beyond numeric symbols may suggest PSTs' previous mathematics classroom experiences primarily emphasized solving a mathematics problem following certain procedures.

Thus, teacher educators need to provide targeted support to PSTs so that they can reinforce their strengths and improve areas of weaknesses regarding core teaching practices. Teacher educators might ask PSTs to record and represent student written work (Namakshi et al., 2022) and verbal explanations (Sleep & Boerst, 2012) to allow them to not only analyze student mathematical ideas but also reflect their competencies in those practices.

Additionally, teacher education programs need to provide additional support to PSTs so that they can acquire and execute core teaching practices. For example, teacher educators may instruct PSTs to clearly interpret and hypothesize about student thinking processes to fully capture their mathematical thought process, not directly translating their verbatims into numeric-only recordings. Moreover, teacher educators in Korea can ask PSTs to use visual representations to deepen student conceptual understanding (Garcia et al., 2021).

We also suggest that teacher educators use an analytical framework adopted in this study to assess the growth of PSTs in their core teaching practices. The results from assessments can serve as resources for revising the curriculum of teacher education programs to enhance the quality of teaching practices. These efforts also could support the development of PSTs' noticing skills (Jacobs et al., 2010) and MKT (Ball et al., 2009).

We acknowledge a methodological limitation in this study. We relied solely on PSTs' written and drawn responses without follow-up interviews to delve deeper into their thoughts. For example, participating PSTs may not be sure about the extent of freedom they have in order to elucidate student thinking paths. Future studies can include follow-up interviews to enhance the validity and credibility of the findings. Exploring different tasks or topics in subsequent studies could provide further insights into PSTs' skills in recording and representing practices. Additionally, it is important to understand that the results of this study should not be overgeneralized beyond this study's scope. This is because only one site was chosen in each country for the cross-cultural comparative study, and the sample size was relatively small.

Another limitation is the levels of threshold. We classified the performances of PSTs into "outperformed," "moderately performed," and "underperformed." Additionally, we used the 10% difference as a threshold to categorize "minor discrepancies" and "moderate to major discrepancies" between PSTs in the U.S. and

Korea. While this approach that identified the threshold level helped us understand the gap in performance of PSTs, these are relatively subjective criteria. If we used different thresholds, the findings of this study might be different. Thus, readers should be cautious when interpreting our findings and arguments. Future studies can implement statistical analysis (e.g., chi-square test) to analyze PSTs' performances across various components.

We suggest that further studies replicate this research in a more authentic context with real students, allowing PSTs the opportunity to ask follow-up questions to elicit student thinking and thereby create more detailed records. However, even in such a context, PSTs' ability to convey the students' thoughts and intentions requires skillful recording. It should go beyond directly translating students' verbalizations into numeric-only recordings. Bridging the gap between direct verbatim translation and more skillful representation of student mathematical thinking could be achieved by gaining a deeper understanding of how to elicit, interpret, and anticipate student thinking. Also, we recommend conducting further comparative studies involving more sites and participants. Conducting more studies may help uncover more nuanced cultural differences in how teachers record and represent student thinking.

A classroom is a space where numerous communications constantly occur in various forms. The role of recording and representing as an instructional tool has often been overlooked despite its importance. The findings of our study identify which particular aspects of recording and representing show higher or lower performance among the PSTs. This information can be instrumental in developing deliberate training programs in mathematics teacher education. We hope that the prevalent tendencies among PSTs identified in this study will assist teacher educators in devising better ways to support them by addressing the gaps they exhibit.

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